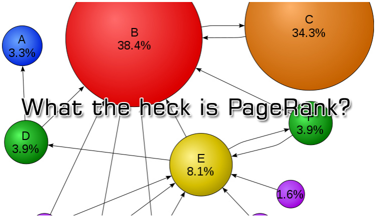
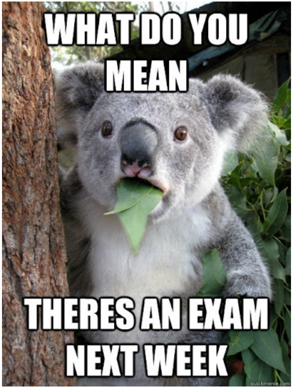
MATH 200C: Linear Algebra



Class 31: Wednesday, April 29, 2026

- PageRank Algorithm
- Notes on Assignment 28
- Useful Facts About the Transpose of a Matrix
- Applications to Markov Chains



Exam 3 Next Wednesday Evening

Predicting the Distant Future

Theorem 10 Stochastic Matrices: If P is a stochastic matrix, then 1 is an eigenvalue of P.

Example: $P = \begin{bmatrix} 4/10 & 7/10 \\ 6/10 & 3/10 \end{bmatrix}$

$$\det\,(P - \lambda I) = \lambda^2 - \tfrac{7}{10}\lambda - \tfrac{3}{10} = (\lambda - 1)(\lambda + \tfrac{3}{10})$$

Finding the eigenvector for P associated with eigenvalue $\lambda = 1$:

$$P - \lambda I = P - I = \begin{bmatrix} 4/10 - 1 & 7/10 \\ 6/10 & 3/10 - 1 \end{bmatrix} = \begin{bmatrix} -6/10 & 7/10 \\ 6/10 & -7/10 \end{bmatrix}$$

Now $P - I$ row reduces to $\begin{bmatrix} -6 & 7 \\ 0 & 0 \end{bmatrix}$ so $(P - I)\mathbf{v} = \mathbf{0}$ becomes

$$-6x + 7y = 0 \text{ where } \mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Thus any vector with $y = \frac{6}{7}x$ is a solution. To find a probability vector which is a solution, we pick the one whose components add to 1.

Thus $1 = x + y = x + \frac{6}{7}x = (1 + \frac{6}{7})x = \frac{13}{7}x$.

Thus $x = \frac{7}{13}$ and $y = \frac{6}{13}$.

Theorem 11: If P is an $n \times n$ regular stochastic matrix, then P has a unique steady-state vector $\mathbf{q}$. Further, if $\mathbf{x_0}$ is any initial state and $\mathbf{x_{k+1}} = P\mathbf{x_k}$ for $k = 0, 1, 2, \dots,$ then the Markov chain $\{\mathbf{x_k}\}$ converges to $\mathbf{q}$ as $k \to \infty$

Example: $P = \begin{bmatrix} 4/10 & 7/10 \\ 6/10 & 3/10 \end{bmatrix}$ has

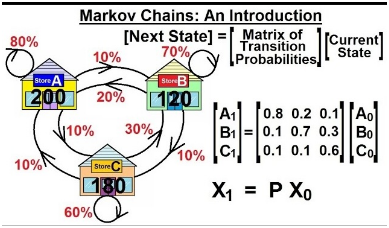
eigenvalue $\lambda_1 = 1$ with eigenvector $\mathbf{v_1} = \begin{bmatrix} 7/13 \\ 6/13 \end{bmatrix}$ and

eigenvalue $\lambda_2 = \frac{-3}{10}$ with eigenvector $\mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Any initial $\mathbf{x_0}$ has the form $\begin{bmatrix} a \\ 1-a \end{bmatrix} = 1 \begin{bmatrix} 7/13 \\ 6/13 \end{bmatrix} + (\frac{7}{13} - a) \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Then $\mathbf{x_k} = 1(1)^k \begin{bmatrix} 7/13 \\ 6/13 \end{bmatrix} + (\frac{7}{13} - a) \left(\frac{-3}{10}\right)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 7/13 \\ 6/13 \end{bmatrix} = \mathbf{q}$

Theorem 11: If P is an $n \times n$ regular stochastic matrix, then P has a unique steady-state vector $\mathbf{q}$. Further, if $\mathbf{x_0}$ is any initial state and $\mathbf{x_{k+1}} = P\mathbf{x_k}$ for $k = 0, 1, 2, \dots,$ then the Markov chain $\{\mathbf{x_k}\}$ converges to $\mathbf{q}$ as $k \to \infty$



$$A = \begin{bmatrix} 8/10 & 2/10 & 1/10 \\ 1/10 & 7/10 & 3/10 \\ 1/10 & 1/10 & 6/10 \end{bmatrix}$$

Characteristic Polynomial

$$\det\,(A - \lambda I) = (1/10)(\lambda - 1)(5\lambda - 3)(2\lambda - 1)$$

λ_1	Eigenvector	λ_2	Eigenvector	λ_3	Eigenvector
1	$\begin{bmatrix} 9/20 \\ 7/20 \\ 4/20 \end{bmatrix}$	3/5	$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$	1/2	$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$$\mathbf{x_k} \to \mathbf{q} = \begin{bmatrix} 9/20 \\ 7/20 \\ 4/20 \end{bmatrix} = \begin{bmatrix} .45 \\ .35 \\ .20 \end{bmatrix}$$

Larry Page and Sergey Brin

Google PageRank Algorithm

Definitions:A **graph** is a collection of points (**vertices**) and lines (**edges**) connecting some of the points.
A **random walk** on a graph is a Markov Chain where at each step the chain is equally likely to move along any of the edges attached to the vertex.

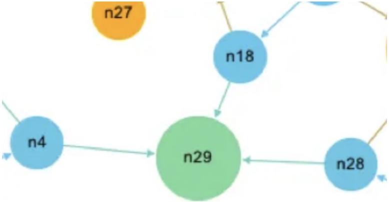
A **directed graph** is a graph in which the vertices are joined not by lines but by arrows.

A **simple random walk** on a directed graph allows the chain to move from vertex to vertex but only in the directions allowed by the arrows.

A **dangling node** is a vertex from which no arrow leads out.

Is there a dangling node?

A **dangling node** is a vertex from which no arrow leads out.



Node n29 is dangling.

Google models the Web as a directed graph: vertices are pages and an arrow goes from page j to page i if there is a hyperlink from page j to page i.

The PageRank Algorithm is a simple random walk on this directed graph modified so that the transition matrix is regular.

Definition: If P is a stochastic matrix, then a **steady-state vector** (or **equilibrium vector**) for P is a probability vector $\mathbf{q}$ so that that $P\mathbf{q} = \mathbf{q}$. If some positive power P^k of P contains only strictly positive entries, then P is called **regular**.

Theorem: If P is a regular $n \times n$ transition matrix with $n \geq 2$, then the following are all true:

- There is a stochastic matrix $\Pi = \lim_{m\to\infty} P^m$
- Each column of Π is the same probability vector $\mathbf{q}$.
- For any initial probability vector $\mathbf{x_0}$, we have $\lim_{m\to\infty} P^m\mathbf{x_0} = \mathbf{q}$.
- The vector $\mathbf{q}$ is the unique probability vector that is an eigenvector of P associated with the eigenvalue 1.
- All other eigenvalues λ of P have $|\lambda| < 1$.

Google models the Web as a directed graph: vertices are pages and an arrow goes from page j to page i if there is a hyperlink from page j to page i.

The PageRank Algorithm is a simple random walk on this directed graph modified so that the transition matrix is regular.

Adjustment 1: If the surfer reaches a dangling node, then the surfer picks any web page with equal probability and moves to that page:

If j is an absorbing state for P an $n \times n$ matrix , then replace column j of P with the vector all of whose entries are $1/n$ to create a new matrix P^*

Adjustment 2: Let p be a number between 0 and 1.
Assume the surfer is now at page j. With probability p, the surfer picks from among all possible links from page with equal probability and moves to that page.
With probability $1 - p$, the surfer picks any page in the Web with equal probability and moves to that page; that is, the **Google matrix** is the new transition matrix G where

$$G = pP^* + (1-p)K$$

where K is an $n \times n$ matrix all of whose entries are $1/n$.

$$
\begin{array}{cc}
\textit{From} & \\
\textit{IOS} \ \textit{Android} & \textit{To} \\
\begin{bmatrix} .70 & .15 \\ .30 & .85 \end{bmatrix} & \begin{array}{c} \textit{IOS} \\ \textit{Android} \end{array}
\end{array}
$$